

Ergodic properties of invariant measures for $C^{1+\alpha}$ non-uniformly hyperbolic systems

Chao Liang*, Wenxiang Sun[†] and Xueting Tian[‡]

* Applied Mathematical Department, The Central University of Finance and Economics,
Beijing 100081, China

(chaol@cufe.edu.cn)

[†] LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China
(sunwx@math.pku.edu.cn)

[‡] Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing
100190, China

& School of Mathematical Sciences, Peking University, Beijing 100871, China
(tianxt@amss.ac.cn & txt@pku.edu.cn)

Abstract

For an ergodic hyperbolic measure ω of a $C^{1+\alpha}$ diffeomorphism, there is a ω full-measured set $\tilde{\Lambda}$ such that every nonempty, compact and connected subset V of $\mathcal{M}_{inv}(\tilde{\Lambda})$ coincides with the accumulating set of time averages of Dirac measures supported at *one orbit*, where $\mathcal{M}_{inv}(\tilde{\Lambda})$ denotes the space of invariant measures supported on $\tilde{\Lambda}$. Such state points corresponding to a fixed V are dense in the support $supp(\omega)$. Moreover $\mathcal{M}_{inv}(\tilde{\Lambda})$ can be accumulated by time averages of Dirac measures supported at *one orbit*, and such state points form a residual subset of $supp(\omega)$. These extend results of Sigmund [9] from uniformly hyperbolic case to non-uniformly hyperbolic case. As a corollary, irregular points form a residual set of $supp(\omega)$.

1 Introduction

Sigmund [9] in 1970 invented two approximation properties for C^1 uniformly hyperbolic diffeomorphisms: one is that invariant measures can be approximated by periodic measures, the other is that every nonempty, compact and connected subset of the space of invariant measures coincides with the accumulating set of time averages of Dirac measures supported at one orbit and such orbits are dense. The first approximation property had realized among $C^{1+\alpha}$ non-uniformly hyperbolic diffeomorphisms in 2003, when Hirayama

*Liang is supported by NNSFC(# 10901167 and # 10671006)

[†]Sun is supported by NNSFC(# 10231020) and Doctoral Education Foundation of China

[‡]Tian is the corresponding author

Key words and phrases: Pesin theory, Katok's shadowing lemma, non-uniformly hyperbolic system, specification, maximal oscillation and irregular point

AMS Review: 37C40; 37D25; 37H15; 37A35

[3] proved that periodic measures are dense in the set of invariant measures supported on a full measure set with respect to a hyperbolic *mixing* measure. In 2009, Liang, Liu and Sun [5] replaced the assumption of hyperbolic *mixing* measure by a more natural and weaker assumption of hyperbolic *ergodic* measure and generalized Hirayama's result. The proofs in [3, 5] are both based on Katok's closing and shadowing lemmas of the $C^{1+\alpha}$ Pesin theory. Moreover, the first approximation property is also valid in the C^1 setting with limit domination by using Liao's shadowing lemma for quasi-hyperbolic orbit segments[10].

The specification property for Axiom A systems ensure the two approximation properties in [9]. However, the specification property in a weaker version for non-uniformly hyperbolic systems in [3, 5, 10] is invalid to the second approximation property, though it can deduce the first one. More precisely, to achieve the second approximation property, Sigmund[9] uses the specification property infinitely many times to find the needed orbit. However, for the nonuniformly hyperbolic case, his idea is not suitable: the specification property for *finite* orbit segments in the same Pesin block, introduced in [3, 5, 10], can not be used infinitely many times (even two times), since we can not determine that the given periodic points and the shadowing periodic orbits always stay in the required set $\tilde{\Lambda}$. Therefore, to deal with non-uniformly hyperbolic case, we disinter a new specification property for *infinite* orbit segments (allowing belonging to different Pesin blocks), inspired from Katok's Shadowing Lemma, and use it only once to find the needed orbit and hence avoid induction. Now we start to introduce our results precisely.

Throughout this paper, we consider an $f \in \text{Diff}^{1+\alpha}(M)$ and an ergodic hyperbolic measure ω for f . Let $\Lambda = \cup_{\ell=1}^{\infty} \Lambda_{\ell}$ be the Pesin set associated with ω . We denote by $\omega|_{\Lambda_{\ell}}$ the conditional measure of ω on Λ_{ℓ} . Set $\tilde{\Lambda}_{\ell} = \text{supp}(\omega|_{\Lambda_{\ell}})$ and $\tilde{\Lambda} = \cup_{\ell=1}^{\infty} \tilde{\Lambda}_{\ell}$. Clearly, $f^{\pm 1}\tilde{\Lambda}_{\ell} \subset \tilde{\Lambda}_{\ell+1}$, and the sub-bundles $E^s(x)$, $E^u(x)$ depend continuously on $x \in \tilde{\Lambda}_{\ell}$. Moreover, $\tilde{\Lambda}$ is f -invariant with ω -full measure.

We denote by $V_f(\nu)$ the set of accumulation measures of time averages

$$\nu^N = \frac{1}{N} \sum_{j=0}^{N-1} f_*^j \nu.$$

Then $V_f(\nu)$ is a nonempty, closed and connected subset of $\mathcal{M}_{inv}(M)$. And we denote by $V_f(x)$ the set of accumulation measures of time averages

$$\nu^N = \frac{1}{N} \sum_{j=0}^{N-1} \delta_{f^j x},$$

where δ_x denotes the Dirac measure at x . Now we state our main theorems as follows.

Theorem 1.1. *For every nonempty connected set $V \subseteq \{\nu \in \mathcal{M}_{inv}(M) \mid \nu(\tilde{\Lambda}) = 1\}$, there exists a point $x \in M$ such that*

$$\text{Closure}(V) = V_f(x).$$

Moreover, the set of such x is dense in $\text{supp}(\omega)$, that is, the closure of this set contains $\text{supp}(\omega)$.

A point $x \in M$ is called to be a *generic point* for an f -invariant measure ν if for any $\phi \in C^0(M, \mathbb{R})$, the limit $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x)$ exists and is equal to $\int \phi d\nu$. As a corollary of Theorem 3.1, the following holds.

Corollary 1.2. *Every f -invariant measure supported on $\tilde{\Lambda}$ has generic points and all generic points form a dense subset in $\text{supp}(\omega)$.*

A point $x \in M$ is said to have *maximal oscillation* if

$$V_f(x) \supseteq \text{Closure}\{\nu \in \mathcal{M}_{\text{inv}}(M) \mid \nu(\tilde{\Lambda}) = 1\}.$$

We can deduce from Theorem 1.1 that the points having maximal oscillation are dense in $\text{supp}(\omega)$. As an extension to Theorem 1.1, we go on to prove that they form a residual subset of $\text{supp}(\omega)$.

Theorem 1.3. *The set of points having maximal oscillation is residual in $\text{supp}(\omega)$.*

Remark 1.4. For any homeomorphism $f : X \rightarrow X$ on a compact metric space preserving an ergodic measure ω , if (f, ω) has specification property (see Theorem 3.1 for more details), analogous arguments and results as in Theorem 1.1 and 1.3 are adaptable. \square

A point is called to be an *irregular point* if there is a continuous function $\phi \in C^0(M, \mathbb{R})$, such that the limit $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x)$ does not exist. As an application of Theorem 1.3, we have the below result.

Theorem 1.5. *If $\text{Closure}(\mathcal{M}_{\text{inv}}(\tilde{\Lambda}))$ is nontrivial (i.e., contains at least one measure different from ω), then the set of all irregular points is residual in $\text{supp}(\omega)$.*

This paper is organized as follows. In section 2, we recall the definition of Pesin set and Katok's shadowing lemma. In section 3, we develop a new specification property and verify that (f, ω) admits this property. In section 4, we use the information on orbit segments to describe that of an invariant measure. In section 5 we use the results in section 3 and 4 to prove Theorem 1.1 and then in section 6 we use Theorem 1.1 to prove Theorem 1.3 and 1.5.

2 Preliminaries

We recall the concept of Pesin set and recall some preliminary lemmas in this section.

2.1 Pesin set ([4, 7])

Given $\lambda, \mu \gg \varepsilon > 0$, and for all $k \in \mathbb{Z}^+$, we define $\Lambda_k = \Lambda_k(\lambda, \mu; \varepsilon)$ to be all points $x \in M$ for which there is a splitting $T_x M = E_x^s \oplus E_x^u$ with invariant property $D_x f^m(E_x^s) = E_{f^m x}^s$ and $D_x f^m(E_x^u) = E_{f^m x}^u$ satisfying:

- (a) $\|Df^n|_{E_{f^m x}^s}\| \leq e^{\varepsilon k} e^{-(\lambda - \varepsilon)n} e^{\varepsilon|m|}, \forall m \in \mathbb{Z}, n \geq 1;$
- (b) $\|Df^{-n}|_{E_{f^m x}^u}\| \leq e^{\varepsilon k} e^{-(\mu - \varepsilon)n} e^{\varepsilon|m|}, \forall m \in \mathbb{Z}, n \geq 1;$
- (c) $\tan(\angle(E_{f^m x}^s, E_{f^m x}^u)) \geq e^{-\varepsilon k} e^{-\varepsilon|m|}, \forall m \in \mathbb{Z}.$

We set $\Lambda = \Lambda(\lambda, \mu; \varepsilon) = \bigcup_{k=1}^{+\infty} \Lambda_k$ and call Λ a Pesin set.

It is obvious that if $\varepsilon_1 < \varepsilon_2$, then $\Lambda(\lambda, \mu; \varepsilon_1) \subseteq \Lambda(\lambda, \mu; \varepsilon_2)$.

According to Oseledec Theorem, ω has s ($s \leq d = \dim M$) nonzero Lyapunov exponents

$$\lambda_1 < \dots < \lambda_r < 0 < \lambda_{r+1} < \dots < \lambda_s$$

with associated Oseledec splitting

$$T_x M = E_x^1 \oplus \dots \oplus E_x^s, \quad x \in O(\omega),$$

where we recall that $O(\omega)$ denotes an Oseledec basin of ω . If we denote by λ the absolute value of the largest negative Lyapunov exponent λ_r and μ the smallest positive Lyapunov exponent λ_{r+1} and set $E_x^s = E_x^1 \oplus \dots \oplus E_x^r$, $E_x^u = E_x^{r+1} \oplus \dots \oplus E_x^s$, then we get a Pesin set $\Lambda = \Lambda(\lambda, \mu; \varepsilon)$ for a small ε . We call it the Pesin set associated with ω . It follows (see, for example, Proposition 4.2 in [7]) that $\omega(\Lambda \setminus O(\omega)) + \omega(O(\omega) \setminus \Lambda) = 0$.

The following statements are elementary:

- (a) $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \subseteq \dots;$
- (b) $f(\Lambda_k) \subseteq \Lambda_{k+1}, \quad f^{-1}(\Lambda_k) \subseteq \Lambda_{k+1};$
- (c) Λ_k is compact for $\forall k \geq 1;$
- (d) for $\forall k \geq 1$ the splitting $x \rightarrow E_x^u \oplus E_x^s$ depends continuously on $x \in \Lambda_k$.

2.2 Shadowing lemma

Let $(\delta_k)_{k=1}^{+\infty}$ be a sequence of positive real numbers. Let $(x_n)_{n=-\infty}^{+\infty}$ be a sequence of points in $\Lambda = \Lambda(\lambda, \mu, \varepsilon)$ for which there exists a sequence $(s_n)_{n=-\infty}^{+\infty}$ of positive integers satisfying:

- (a) $x_n \in \Lambda_{s_n}, \quad \forall n \in \mathbb{Z};$
- (b) $|s_n - s_{n-1}| \leq 1, \quad \forall n \in \mathbb{Z};$

$$(c) \quad d(fx_n, x_{n+1}) \leq \delta_{s_n}, \quad \forall n \in \mathbb{Z};$$

then we call $(x_n)_{n=-\infty}^{+\infty}$ a $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit. Given $\eta > 0$, a point $x \in M$ is an η -shadowing point for the $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit if $d(f^n x, x_{n+1}) \leq \eta \varepsilon_{s_n}$, $\forall n \in \mathbb{Z}$, where $\varepsilon_k = \varepsilon_0 e^{-\varepsilon k}$ and ε_0 is a constant.

Lemma 2.1. (*Shadowing lemma [4, 7]*) *Let $f : M \rightarrow M$ be a $C^{1+\alpha}$ diffeomorphism, with a non-empty Pesin set $\Lambda = \Lambda(\lambda, \mu; \varepsilon)$ and fixed parameters, $\lambda, \mu \gg \varepsilon > 0$. For $\forall \eta > 0$ there exists a sequence $(\delta_k)_{k=1}^{+\infty}$ such that for any $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit there exists a unique η -shadowing point.*

3 Specification Property for Non-uniformly hyperbolic systems

In this section, we develop a new specification property for $C^{1+\alpha}$ non-uniformly hyperbolic systems, which will play crucial role in the proof of Theorem 1.1.

Theorem 3.1. *(f, ω) has specification property in the following sense. For any $\eta > 0$ there is a sequence of integers $\{M_{k,l} = M_{k,l}(\eta)\}_{k,l \geq 1}$ satisfying:*

Given a sequence of integers $\{k_s \mid k_s \geq 1\}_{s \in [a,b] \cap \mathbb{Z}}$ for any $-\infty \leq a < b \leq \infty$ and a sequence of orbit segments $\{\{f^i(x_s)\}_{i=0}^{n_s} \mid x_s, f^{n_s}x_s \in \tilde{\Lambda}_{k_s}, n_s \in \mathbb{N}\}_{s \in [a,b] \cap \mathbb{Z}}$, there exist a shadowing point $z \in M$ and an increasing sequence of integers $\{c_s\}_{s \in [a-1,b] \cap \mathbb{Z}}$ with $0 \leq c_{s+1} - c_s - n_{s+1} \leq M_{k_s, k_{s+1}}$ ($s \in [a-1, b-1] \cap \mathbb{Z}$) such that

$$d(f^{c_{s-1}+j}z, f^j x_s) < \eta \varepsilon_{k_{s+1}}, \quad \forall j = 0, 1, \dots, n_s - 1, \quad s \in [a, b] \cap \mathbb{Z},$$

where $\varepsilon_k = \varepsilon_0 e^{-\varepsilon k}$ and ε_0 is a constant.

In particular, if a and b are finite integers, the shadowing point z should be periodic with period $\pi = c_b - c_{a-1}$.

Remark 3.2. The consequence of Theorem 3.4[5] or Hirayama's definition for specification property is a particular case of the above theorem. More precisely, they considered *finite* orbit segments and asked the beginning and ending points of these segments must be in the *same* block $\tilde{\Lambda}_l$.

Proof of Theorem 3.1

For $\forall \eta > 0$, by Lemma 2.1 there exists a sequence $(\delta_k)_{k=1}^{+\infty}$ such that for any $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit there exists a unique η -shadowing point.

Let k_* big enough such that $\omega(\tilde{\Lambda}_k) > 0$ for all $k \geq k_*$. For every $k \geq k_*$, take and fix for $\tilde{\Lambda}_k$ a finite cover $\alpha_k = \{V_1^k, V_2^k, \dots, V_{r_k}^k\}$ by nonempty open balls V_i^k in M such

that $\text{diam}(U_i^k) < \delta_{k+1}$ and $\omega(U_i^k) > 0$ where $U_i^k = V_i^k \cap \tilde{\Lambda}_k$, $i = 1, 2, \dots, r_k$. Since ω is f -ergodic, by Birkhoff ergodic theorem we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{h=0}^{n-1} \omega(f^{-h}(U_i^\ell) \cap U_j^k) = \omega(U_i^\ell) \omega(U_j^k) > 0, \quad (3.1)$$

$\forall k, \ell \geq k_*$, $\forall 1 \leq i \leq r_\ell, 1 \leq j \leq r_k$. Then take

$$X_{i,j}^{k,\ell} = \min\{h \in \mathbb{N} \mid h \geq 1, \omega(f^{-h}(U_i^\ell) \cap U_j^k) > 0\}. \quad (3.2)$$

By (3.1), $1 \leq X_{i,j}^{k,\ell} < +\infty$. Let

$$M_{k,\ell} = \max_{1 \leq i \leq r_k, 1 \leq j \leq r_\ell} X_{i,j}^{k,\ell}.$$

Now let us consider an increasing sequence of integers $\{k_s \mid k_s \geq k_*\}_{s \in \mathbb{Z}}$ and a sequence of orbit segments $\{\{f^i(x_s)\}_{i=0}^{n_s} \mid x_s, f^{n_s}x_s \in \tilde{\Lambda}_{k_s}, n_s \in \mathbb{N}\}_{s \in \mathbb{Z}}$. For each $s \in \mathbb{Z}$, we take and fix two integers s_0 and s_1 so that

$$x_s \in U_{s_0}^{k_s}, f^{n_s}x_s \in U_{s_1}^{k_s}, s \in \mathbb{Z}.$$

Take $y_s \in U_{s_1}^{k_s}$ by (3.2) such that $f^{X_{(s+1)_0, s_1}^{k_s, l}} y_s \in U_{(s+1)_0}^{k_{s+1}}$ for $s \in \mathbb{Z}$. Thus we get a $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit in M :

$$\dots \{f^t(x_1)\}_{t=0}^{n_1} \cup \{f^t(y_1)\}_{t=0}^{X_{20, 11}^{k_1, k_2}} \cup \{f^t(x_2)\}_{t=0}^{n_2} \cup \{f^t(y_2)\}_{t=0}^{X_{30, 21}^{k_2, k_3}} \cup \dots$$

More precisely,

$$x_s, f^{n_s}(x_s) \in \tilde{\Lambda}_{k_s} \subseteq \Lambda_{k_s}, y_s \in \tilde{\Lambda}_{k_s} \subseteq \Lambda_{k_s} \text{ and } f^{X_{(s+1)_0, s_1}^{k_s, k_{s+1}}} y_s \in \tilde{\Lambda}_{k_{s+1}} \subseteq \Lambda_{k_{s+1}},$$

and

$$d(f^{n_s}(x_s), y_s) < \delta_{k_{s+1}}, d(f^{X_{(s+1)_0, s_1}^{k_s, k_{s+1}}} y_s, x_{s+1}) < \delta_{k_{s+1}+1}, \forall s \in \mathbb{Z}.$$

Hence there exists an η -shadowing point $z \in M$ such that

$$d(f^{c_{s-1}+j} z, f^j x_s) < \eta \varepsilon_{k_{s+1}}, \forall j = 0, 1, \dots, n_s - 1, s \in \mathbb{Z},$$

where

$$c_s = \begin{cases} 0, & \text{for } s = 0 \\ \sum_{j=0}^{s-1} [n_j + X_{(j+1)_0, j_1}^{k_j, k_{j+1}}], & \text{for } s > 0 \\ -\sum_{j=s}^{-1} [n_j + X_{(j+1)_0, j_1}^{k_j, k_{j+1}}], & \text{for } s < 0. \end{cases}$$

This ends the proof. \square

4 Characterizing invariant measures by orbit segments

It is well-known that for ergodic systems, the time average is the same for almost all initial points and coincides with the space average due to Birkhoff Ergodic Theorem. However, it is not true for general measure-preserving systems (for example, the measure supported on two periodic orbits). Inspired by Ergodic Decomposition Theorem, we prove in the following that the space average can be approximated by the information along finite orbit segments.

Given a finite subset $F \subseteq C^0(M, \mathbb{R})$, we denote

$$\|F\| = \max\{\|\xi\|; \xi \in F\}.$$

Proposition 4.1. *Suppose $f : X \rightarrow X$ is a homeomorphism on a compact metric space and ν is an f -invariant measure. Then for any numbers $\varepsilon > 0$, any finite subset $F \subseteq C^0(M, \mathbb{R})$ and any set $\Delta \subseteq X$ with $\nu(\Delta) > (1 + \frac{\varepsilon}{16\|F\|})^{-1}$, there are a measurable partition $\{R_j\}_{j=1}^b$ of Δ , ($b \in \mathbb{Z}$) and a positive integer T , such that for any $x_j \in R_j$ and any integers $T_j \geq T$, ($1 \leq j \leq b$), we have that*

$$|\int \xi(x) d\nu - \sum_{j=1}^b \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j))| < \varepsilon, \forall \xi \in F,$$

for any $\theta_j > 0$ satisfying $|\theta_j - \frac{\nu(R_j)}{\nu(\Delta)}| < \frac{\varepsilon}{2b\|F\|}$, $1 \leq j \leq b$.

Proof Let $A = \sup\{|\xi^*(x)| \mid x \in Q(f), \xi \in F\}$. Denote by $[a]$ the maximal integer not exceeding a . For $j = 1, \dots, [\frac{32A\|F\|}{\varepsilon\|\xi\|}] + 1$, $\xi \in F$, set $Q_j(\xi) = \{x \in Q(f) \mid -A + \frac{(j-1)\varepsilon}{16\|F\|} \|\xi\| \leq \xi^*(x) < -A + \frac{j\varepsilon}{16\|F\|} \|\xi\|\}$. Let $\mathcal{B} = \bigvee_{\xi \in F} \{Q_1(\xi), \dots, Q_{[\frac{32A\|F\|}{\varepsilon\|\xi\|}] + 1}(\xi)\}$, where $\alpha \vee \beta = \{A_i \cap B_j \mid A_i \in \alpha, B_j \in \beta\}$ for partitions $\alpha = \{A_i\}$, $\beta = \{B_j\}$. Then $\mathcal{B} = \{R_j\}_{j=1}^b$ is a partition of $Q(f)$. Hence that the positive-measure sets in $\{R_j \cap \Delta\}_{j=1}^b$ form a partition of Δ . For simplicity, we still denote this partition by $\mathcal{B} = \{R_j\}_{j=1}^b$. Then by the definition of \mathcal{B} and $Q_j(\xi)$ above, we have

$$\begin{aligned}
& \left| \int_{\Delta} \xi^*(x) d\nu - \sum_{j=1}^b \theta_j \xi^*(x_j) \right| \\
&= \left| \int_{\Delta} \xi^*(x) d\nu - \sum_{j=1}^b \frac{\nu(R_j)}{\nu(\Delta)} \xi^*(x_j) \right| + \left| \sum_{j=1}^b \frac{\nu(R_j)}{\nu(\Delta)} \xi^*(x_j) - \sum_{j=1}^b \theta_j \xi^*(x_j) \right| \\
&\leq \left| \int_{\Delta} \xi^*(x) d\nu - \sum_{j=1}^b \nu(R_j) \xi^*(x_j) \right| + \left| \sum_{j=1}^b \nu(R_j) \xi^*(x_j) - \sum_{j=1}^b \frac{\nu(R_j)}{\nu(\Delta)} \xi^*(x_j) \right| \\
&+ \left| \sum_{j=1}^b \left(\frac{\nu(R_j)}{\nu(\Delta)} - \theta_j \right) \xi^*(x_j) \right| \\
&\leq \sum_{j=1}^b \nu(R_j) \max_{y \in R_j} |\xi^*(y) - \xi^*(x_j)| + \left| \sum_{j=1}^b \nu(R_j) \xi^*(x_j) \right| \cdot \left(\frac{1}{\nu(\Delta)} - 1 \right) + \frac{\varepsilon}{2b\|F\|} \sum_{j=1}^b |\xi^*(x_j)| \\
&\leq \frac{1}{8\|F\|} \cdot \varepsilon \|\xi\| \cdot \sum_{j=1}^b \nu(R_j) + A \cdot \sum_{j=1}^b \nu(R_j) \cdot \frac{\varepsilon}{16\|F\|} + \frac{\varepsilon A}{2\|F\|} \\
&\leq \frac{1}{8\|F\|} \cdot \varepsilon \|\xi\| + \frac{\varepsilon A}{16\|F\|} + \frac{\varepsilon A}{2\|F\|} \\
&\leq \frac{11\varepsilon}{16}, \quad \xi \in F.
\end{aligned}$$

For the last inequality, note that $A \leq \|F\|$.

On the other hand, we shall take T large enough such that for all $T_j \geq T$,

$$|\xi^*(x_j) - \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j))| < \frac{\varepsilon}{16}, \quad \forall j = 1, 2, \dots, b, \quad \xi \in F.$$

Thus

$$\begin{aligned}
& \left| \int_{\Delta} \xi^*(x) d\nu - \sum_{j=1}^b \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right| \\
&\leq \left| \int_{\Delta} \xi^*(x) d\nu - \sum_{j=1}^b \theta_j \xi^*(x_j) \right| \\
&+ \left| \sum_{j=1}^b \theta_j \xi^*(x_j) - \sum_{j=1}^b \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right| \\
&< \frac{11\varepsilon}{16} + \left| \sum_{j=1}^b \theta_j \left(\xi^*(x_j) - \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right) \right| \\
&< \frac{11\varepsilon}{16} + \sum_{j=1}^b \theta_j \frac{\varepsilon}{16} \leq \frac{3\varepsilon}{4}, \quad \xi \in F.
\end{aligned}$$

Note that $\nu(\Delta) > (1 + \frac{\varepsilon}{16\|F\|})^{-1} > 1 - \frac{\varepsilon}{16\|F\|}$. Hence,

$$\begin{aligned}
& \left| \int_M \xi(x) d\nu - \sum_{j=1}^b \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right| \\
&= \left| \int_M \xi^*(x) d\nu - \sum_{j=1}^b \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right| \\
&\leq \left| \int_M \xi(x) d\nu - \int_\Delta \xi(x) d\nu \right| + \left| \int_\Delta \xi^*(x) d\nu - \sum_{j=1}^b \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right| \\
&\leq \|\xi\| \cdot (1 - \nu(\Delta)) + \frac{\varepsilon}{4} \\
&\leq \|\xi\| \frac{\varepsilon}{16\|F\|} + \frac{3\varepsilon}{4} \\
&< \varepsilon, \quad \xi \in F.
\end{aligned}$$

This ends the proof. \square

The following lemma is Lemma 3.7 in [5].

Lemma 4.2. *Let $f : X \rightarrow X$ be a homeomorphism of a compact metric space preserving an ergodic measure ω . Let $\Gamma_j \subset X$ be measurable sets with $\omega(\Gamma_j) > 0$ and for $x \in \Gamma_j$, let*

$$S(x, \Gamma_j) := \{r \in \mathbb{N} \mid f^r x \in \Gamma_j\},$$

$j = 1, \dots, k$. Take $1 > \gamma > 0$, $T \geq 1$. Then for ω -a.e. $x_j \in \Gamma_j$ there exists $n_j = n_j(x_j) \in S(x_j, \Gamma_j)$ such that $n_j \geq T$ and

$$0 < \frac{|n_1 - n_j| + \dots + |n_{j-1} - n_j| + |n_{j+1} - n_j| + \dots + |n_k - n_j|}{\sum_{j=1}^k n_j} < \gamma,$$

where $j = 1, \dots, k$.

Proposition 4.3. *Suppose $f : X \rightarrow X$ is a homeomorphism on a compact metric space and ν is an f -invariant measure. Then for any numbers $\varepsilon > 0$, any finite subset $F \subseteq C^0(M, \mathbb{R})$ and any set $\Delta \subseteq X$ with $\nu(\Delta) > (1 + \frac{\varepsilon}{16\|F\|})^{-1}$, there is a measurable partition $\{R_j\}_{j=1}^b$ of Δ , ($b \in \mathbb{Z}$) such that for any positive integer T , and any recurrence points $x_j \in R_j$, there exist recurrence times $T_j \geq T$, ($1 \leq j \leq b$) satisfying*

$$\left| \int \xi(x) d\nu - \frac{1}{\sum_{j=1}^b \theta_j T_j} \sum_{j=1}^b \theta_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right| < \varepsilon, \forall \xi \in F,$$

for any $\theta_j > 0$ satisfying $|\theta_j - \frac{\nu(R_j)}{\nu(\Delta)}| < \frac{\varepsilon}{2b\|F\|}$, $1 \leq j \leq b$.

Proof Take the same partition $\{R_j\}_{j=1}^b$ as in Proposition 4.1. By Poincaré's Recurrence Lemma, ν -almost every points in R_j are recurrence points. Fix a sequence of recurrence points $x_j \in R_j$ and denote their recurrence time by T_j . By the finiteness of F and Lemma 4.2, we can choose integers $T_j \geq T$ for any $T > 0$ such that

$$\left| \frac{\theta_1(T_1 - T_j) + \cdots + \theta_{j-1}(T_{j-1} - T_j) + \theta_{j+1}(T_{j+1} - T_j) + \cdots + \theta_b(T_b - T_j)}{\sum_{i=1}^b \theta_i T_i} \right| < \frac{\varepsilon}{16\|F\|}$$

for $j = 1, 2, \dots, b$. Thus

$$\begin{aligned} & \left| \sum_{j=1}^b \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j} \xi(f^h(x_j)) - \frac{1}{\sum_{i=1}^b \theta_i T_i} \sum_{j=1}^b \theta_j \sum_{h=0}^{T_j} \xi(f^h(x_j)) \right| \\ &= \left| \sum_{j=1}^b \frac{\theta_j}{(\theta_1 + \cdots + \theta_b)} \frac{1}{T_j} \sum_{h=0}^{T_j} \xi(f^h(x_j)) - \sum_{j=1}^b \frac{\theta_j}{\theta_1 T_1 + \cdots + \theta_b T_b} \sum_{h=0}^{T_j} \xi(f^h(x_j)) \right| \\ &= \left| \sum_{j=1}^b \theta_j \frac{\theta_1(T_1 - T_j) + \cdots + \theta_{j-1}(T_{j-1} - T_j) + \theta_{j+1}(T_{j+1} - T_j) + \cdots + \theta_b(T_b - T_j)}{\sum_{i=1}^b \theta_i T_i} \right. \\ & \quad \cdot \left. \frac{1}{T_j} \sum_{h=0}^{T_j} \xi(f^h(x_j)) \right| \\ &\leq \left| \sum_{j=1}^b \theta_j \frac{\theta_1(T_1 - T_j) + \cdots + \theta_{j-1}(T_{j-1} - T_j) + \theta_{j+1}(T_{j+1} - T_j) + \cdots + \theta_b(T_b - T_j)}{\sum_{i=1}^b \theta_i T_i} \right| \cdot \|\xi\| \\ &\leq \sum_{j=1}^b \theta_j \left| \frac{\theta_1(T_1 - T_j) + \cdots + \theta_{j-1}(T_{j-1} - T_j) + \theta_{j+1}(T_{j+1} - T_j) + \cdots + \theta_b(T_b - T_j)}{\sum_{i=1}^b \theta_i T_i} \right| \cdot \|\xi\| \\ &\leq \sum_{j=1}^b \theta_j \frac{\varepsilon}{16\|F\|} \|\xi\| \\ &\leq \frac{\varepsilon}{16}, \quad \xi \in F. \end{aligned} \tag{4.3}$$

Note that $\nu(\Delta) > (1 + \frac{\varepsilon}{16\|F\|})^{-1} > 1 - \frac{\varepsilon}{16\|F\|}$. Combining with Proposition 4.1 and inequality (4.3), one deduces that

$$\begin{aligned} & \left| \int_M \xi(x) d\nu - \frac{1}{\sum_{i=1}^b \theta_i T_i} \sum_{j=1}^b \theta_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right| \\ &\leq \left| \int_M \xi(x) d\nu - \sum_{j=1}^b \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right| \\ &+ \left| \sum_{j=1}^b \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) - \frac{1}{\sum_{i=1}^b \theta_i T_i} \sum_{j=1}^b \theta_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right| \\ &\leq \varepsilon + \frac{\varepsilon}{16} \\ &< 2\varepsilon, \quad \xi \in F. \end{aligned}$$

Hence we complete the proof. \square

Remark 4.4. Through the proof of the previous proposition, one can obtain that the conclusion is suitable for any finer partition of $\{R_j\}_{j=1}^b$. \square

Proposition 4.5. *Let ν be an f -invariant measure supported on $\tilde{\Lambda}$. Then for any numbers $\zeta, \delta > 0$ and any finite subset $F \subseteq C^0(M, \mathbb{R})$, there are a number $k_\nu \in \mathbb{Z}^+$ and orbit segments $\{z_j, fz_j, \dots, f^{n_j-1}z_j\}_{j=1}^b$ with $z_j, f^{n_j}z_j \in \tilde{\Lambda}_{k_\nu}$ and $d(f^{n_j}z_j, z_{j+1}) < \delta$, $j = 1, \dots, b-1$, satisfying that*

$$\left| \int \xi(x) d\nu - \frac{1}{\sum_{j=1}^b n_j} \sum_{j=1}^b \sum_{h=0}^{n_j-1} \xi(f^h(z_j)) \right| < \zeta, \forall \xi \in F.$$

Proof Take k_ν large such that $\nu(\tilde{\Lambda}_{k_\nu}) > (1 + \frac{\zeta}{16\|F\|})^{-1}$. Applying Proposition 4.3 with $\Delta = \tilde{\Lambda}_{k_\nu}$ and $\varepsilon = \zeta$, we obtain a finite partition $\{R_j\}_{j=1}^b$ of $\tilde{\Lambda}_{k_\nu}$ with $\text{diam} R_j < \delta$ and recurrence points $x_j \in R_j$ with large recurrence time T_j , $j = 1, \dots, b$ satisfying that

$$\left| \int \xi(x) d\nu - \frac{1}{\sum_{j=1}^b \theta_j T_j} \sum_{j=1}^b \theta_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right| < \varepsilon, \forall \xi \in F, \quad (4.4)$$

for any $\theta_j > 0$ satisfying $|\theta_j - \frac{\nu(R_j)}{\nu(\tilde{\Lambda}_{k_\nu})}| < \frac{\varepsilon}{2b\|F\|}$, $1 \leq j \leq b$.

Recall that ω is ergodic and thus for any $1 \leq j \leq b$, there is an integer $X_j \geq 1$ such that

$$f^{X_j} R_j \cap R_{j+1} \neq \emptyset, \quad 1 \leq j < b,$$

and

$$f^{X_b} R_b \cap R_1 \neq \emptyset.$$

Take $y_j \in R_j$ so that $f^{X_j} y_j \in R_{j+1}$, $1 \leq j < b$ and $f^{X_b} y_b \in R_1$.

For ζ and b , there exists $S \in \mathbb{N}$ such that for any integer $s > S$, we have $0 < 1/s < \frac{\zeta}{b}$. And then there exists integers $\bar{s}_1, \bar{s}_2, \dots, \bar{s}_b$ satisfying $\bar{s}_j/s \leq \frac{\nu(R_j)}{\nu(\tilde{\Lambda}_{k_\nu})} \leq (\bar{s}_j + 1)/s$. It follows from taking $s_j = \bar{s}_j$ or $\bar{s}_j + 1$ that

$$s = \sum_{j=1}^b s_j \quad \text{and} \quad \left| \frac{\nu(R_j)}{\nu(\tilde{\Lambda}_{k_\nu})} - \frac{s_j}{s} \right| < \frac{\zeta}{2b\|F\|}.$$

Take T_j large enough, such that $\sum_{j=1}^b X_j \ll \sum_{j=1}^b s_j T_j$ and hence it holds that

$$\left| \frac{1}{\sum_{j=1}^b (s_j T_j + X_j)} \sum_{j=1}^b \left(s_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) + \sum_{h=0}^{X_j-1} \xi(f^h(y_j)) \right) - \frac{1}{\sum_{j=1}^b s_j T_j} \sum_{j=1}^b s_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) \right| < \zeta.$$

This inequality combining (4.4) with $\theta_j = \frac{s_j}{s}$ implies that

$$|\int \xi(x)d\nu - \frac{1}{\sum_{j=1}^b (s_j T_j + X_j)} \sum_{j=1}^b (s_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) + \sum_{h=0}^{X_j-1} \xi(f^h(y_j)))| < 3\zeta. \quad (4.5)$$

Let

$$\begin{aligned} z_1 &= \cdots = z_{s_1} = x_1, \quad z_{s_1+1} = y_1, \\ z_{s_1+2} &= \cdots = z_{s_1+s_2+1} = x_2, \quad z_{s_1+s_2+2} = y_2, \\ &\dots\dots\dots \\ z_{\sum_{h=1}^j s_h+j+1} &= \cdots = z_{\sum_{h=1}^{j+1} s_h+j} = x_{j+1} \\ z_{\sum_{h=1}^{j+1} s_h+j+1} &= y_{j+1}, \\ &\dots\dots\dots \\ z_{\sum_{h=1}^{b-1} s_h+b} &= \cdots = z_{\sum_{h=1}^b s_h+b-1} = x_b \\ z_{\sum_{h=1}^b s_h+b} &= y_b. \end{aligned}$$

These $\{z_j\}_{j=1}^{\sum_{h=1}^b s_h+b}$ are the points we want in the proposition and hence we complete the proof. \square

5 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using the specification property developed in section 3 and Proposition 4.5 in section 4.

Proof of Theorem 1.1 If $\{\varphi_j\}_{j=1}^\infty$ is a dense subset of $C^0(M, \mathbb{R})$, then

$$\tilde{d}(\nu, m) = \sum_{j=1}^\infty \frac{|\int \varphi_j d\nu - \int \varphi_j dm|}{2^j \|\varphi_j\|}$$

is a metric on $\mathcal{M}(M)$ giving the weak* topology, see e.g. [11]. It is well known that $\mathcal{M}_{inv}(M)$ is a compact metric subspace of $\mathcal{M}(M)$ in the weak* topology. For any nonempty closed connected set $V \subseteq \{\nu \in \mathcal{M}_{inv}(M) | \nu(\tilde{\Lambda}) = 1\}$, there exists a sequence of closed balls B_n in $\mathcal{M}_{inv}(M)$ with radius ζ_n in the metric \tilde{d} with the weak* topology such that the following holds:

- (a) $B_n \cap B_{n+1} \cap V \neq \emptyset$,
- (b) $\cap_{N=1}^\infty \cup_{n \geq N} B_n = \text{Closure}(V)$,
- (c) $\lim_{n \rightarrow +\infty} \zeta_n = 0$.

By (a), we take $Y_n \in B_n \cap V$.

Remark 5.1. In [9], Sigmund assume that Y_n is an atomic measure and thus its information can be characterized by its support(periodic orbit). Hence the remain work is to deal with these periodic orbits by specification property for Axiom A systems. But for our case, we can not directly take Y_n as an atomic measure(even though this is allowed by [5]). The main observation is that the support of these periodic measures may not be contained in $\tilde{\Lambda}$ and therefore, specification property as in Theorem 3.1 becomes invalid. So we emphasis that Y_n must be in V and thus satisfy $Y_n(\tilde{\Lambda}) = 1$. This allows us to choose pseudo-orbits in $\tilde{\Lambda}$ whose information can characterize that of Y_n and for which the specification property is valid. \square

Take a finite set $F_n = \{\varphi_j\}_{j=1}^n \subseteq \{\varphi_j\}_{j=1}^\infty$. Let $x_* \in \tilde{\Lambda}$ be given and for any $\delta > 0$, let U_0 be the open ball of radius δ around x_* . We have to show that there exists an $x \in U_0$ such that $\text{Closure}(V) = V_f(x)$. We divide the following proof into four steps.

Step 1 An estimation of Y_n ($n \geq 1$).

Let $0 < \eta < \frac{\delta}{\varepsilon_0}$ be given and by shadowing lemma we can take and fix $\{\delta_k\}$. Fix $n \in \mathbb{N}$. For ζ_n, F_n , by Proposition 4.5 we choose $k_n = k(Y_n)$ and orbit segments $\{z_j^n, f z_j^n, \dots, f^{n_j-1} z_j^n\}_{j=1}^b$ with $z_j^n, f^{n_j} z_j^n \in \tilde{\Lambda}_{k_n}$ and $d(f^{n_j} z_j^n, z_{j+1}^n) < \delta_{k_{n+1}}$, $j = 1, \dots, b-1$, satisfying that

$$\left| \int \xi(x) dY_n - \frac{1}{\sum_{j=1}^b n_j} \sum_{j=1}^b \sum_{h=0}^{n_j-1} \xi(f^h(z_j^n)) \right| < \zeta_n, \forall \xi \in F_n.$$

Moreover we can take $k_n < k_{n+1}$ for all n .

These segments of orbit segments $\{z_j^n, f z_j^n, \dots, f^{n_j-1} z_j^n\}_{j=1}^b$ form a ‘periodic’ pseudo-orbit. For simplicity, we can assume that the ‘periodic’ pseudo-orbit is composed by one orbit segment $\{x_n, \dots, f^{p_n-1}(x_n)\}$ with $x_n, f^{p_n}(x_n) \in \tilde{\Lambda}_{k_n}$ and $d(x_n, f^{p_n}(x_n)) < \delta_{k_{n+1}}$. Thus, the above inequality can be simplified as

$$\left| \int \xi(x) dY_n - \frac{1}{p_n} \sum_{h=0}^{p_n-1} \xi(f^h(x_n)) \right| < \zeta_n, \forall \xi \in F_n.$$

From this for any m , clearly one has

$$\left| \int \xi(x) dY_n - \frac{1}{m p_n} \sum_{h=0}^{m p_n-1} \xi(f^{h \bmod p_n}(x_n)) \right| < \zeta_n, \forall \xi \in F_n. \quad (5.6)$$

Step 2 Finding a point $\hat{x} \in U_0$ tracing this pseudo-orbit.

Let $M_n = M_{k_{n-1}, k_n}(\eta)$ be numbers defined as in Theorem 3.1. Define

$$\bar{a}_0 = \bar{b}_0 = 0,$$

$$\bar{a}_1 = \bar{b}_0 + M_1, \quad \bar{b}_1 = \bar{a}_1 + 2(\bar{a}_1 + M_2 + p_2)p_1$$

$$\begin{aligned}
\bar{a}_2 &= \bar{b}_1 + M_2, & \bar{b}_1 &= \bar{a}_1 + 2^2(\bar{a}_2 + M_3 + p_3)p_2 \\
&\dots & \dots \\
\bar{a}_n &= \bar{b}_{n-1} + M_n, & \bar{b}_n &= \bar{a}_n + 2^n(\bar{a}_n + M_{n+1} + p_{n+1})p_n \\
&\dots & \dots
\end{aligned}$$

Using Theorem 3.1 and its proof, we can find a point $\hat{x} \in \Lambda$, δ -close to x_* , which η -shadows the orbit segment $\{x_n, \dots, f^{p_n-1}(x_n)\}$ for $m_n = 2^n(\bar{a}_n + M_{n+1} + p_{n+1})$ times for all n and runs from $f^{p_n}x_n$ to x_{n+1} with a time lag of no more than M_{n+1} . More precisely, there exist $\{a_n\}, \{b_n\}$ with

$$a_0 = b_0 = 0,$$

$$b_n = a_n + m_n p_n, \text{ and } a_n - b_{n-1} \leq M_n$$

such that

$$d(f^j \hat{x}, f^{j \bmod p_n} x_n) < \eta \varepsilon_{k_n}, \quad \forall a_n \leq j \leq b_n. \quad (5.7)$$

Remark 5.2. Note that $a_n \leq \bar{a}_n$, $b_n \leq \bar{b}_n$ and

$$b_n - a_n = \bar{b}_n - \bar{a}_n = m_n p_n, \quad a_n - b_{n-1} \leq \bar{a}_n - \bar{b}_{n-1} = M_n.$$

So as $n \rightarrow +\infty$, b_n and a_{n+1} become much larger than a_n, M_{n+1}, p_n and p_{n+1} .

The original technique for Axiom A systems in [9] is not suitable for non-uniformly hyperbolic ones. Sigmund[9] uses the specification property to build inductively a sequence of periodic orbits such that the n -th orbit shadows both the $(n-1)$ -th orbit and the support of the n -th center. In this process the support of the centers and these shadowing periodic orbits are always in the hyperbolic set such that the specification property can be used once by once. Finally, these periodic orbits conjugates to a point \hat{x} . However, for the nonuniform hyperbolic case, Sigmund's idea face a difficulty. That is the specification property can not be used once by once, since we can not predetermine the Pesin block in which the shadowing periodic orbits stay. Therefore, to deal with non-uniformly hyperbolic cases, we disinter a new specification property. More precisely, instead of dealing induction, we construct an infinitely many orbit segments, inspired from Katok's Shadowing Lemma. And we apply this property once and for all to find \hat{x} and hence avoid induction. \square

Step 3 verifying $\text{Closure}(V) \subseteq V_f(\hat{x})$.

Let $\nu \in \text{Closure}(V)$ be given. By (b) and (c) there exists an increasing sequence $n_k \uparrow \rightarrow \infty$ such that

$$Y_{n_k} \rightarrow \nu. \quad (5.8).$$

Let $\xi \in \{\varphi_j\}_{j=1}^\infty = \cup_{n \geq 1} F_n$ be given. Then there is an integer $n_\xi > 0$ such that for any $n \geq n_\xi$, it holds that $\xi \in F_n$. Denote by $w_\xi(\varepsilon)$ the oscillation

$$\max\{\|\xi(y) - \xi(z)\| \mid d(y, z) \leq \varepsilon\}$$

and by ν_n the measure $\delta(\hat{x})^{b_n}$. Thus

$$\int \xi d\nu_n = \frac{1}{b_n} \sum_{j=0}^{b_n-1} \xi(f^j \hat{x}). \quad (5.9)$$

Remark that if A is a finite subset of \mathbb{N} ,

$$\left| \frac{1}{\#A} \sum_{j \in A} \varphi(f^j x) - \frac{1}{\max A + 1} \sum_{j=0}^{\max A} \varphi(f^j x) \right| \leq \frac{2(\max A + 1 - \#A)}{\#A} \|\varphi\| \quad (5.10)$$

for any $x \in M$ and $\varphi \in C^0(M, \mathbb{R})$, where $\#A$ denotes the cardinality of the set A . This inequality (5.10) implies that

$$\left| \frac{1}{b_n - a_n} \sum_{j=a_n}^{b_n} \xi(f^j \hat{x}) - \frac{1}{b_n} \sum_{j=0}^{b_n-1} \xi(f^j \hat{x}) \right| \leq \frac{2a_n}{b_n - a_n} \|\xi\|, \quad \forall n \geq n_\xi. \quad (5.11)$$

On the other hand, combining the inequalities (5.6) and (5.7), one can obtain that

$$\left| \int \xi dY_n - \frac{1}{b_n - a_n} \sum_{j=a_n}^{b_n} \xi(f^j \hat{x}) \right| \leq \zeta_n + w_\xi(\eta \varepsilon_n), \quad \forall n \geq n_\xi. \quad (5.12)$$

Note that

$$\frac{2a_n}{b_n - a_n} \leq \frac{2\bar{a}_n}{\bar{b}_n - \bar{a}_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (5.13)$$

$\zeta_n \rightarrow 0$ due to Remark 5.2 and $w_\xi(\eta \varepsilon_n) \rightarrow 0$ as $n \rightarrow \infty$, it can be deduce by (5.9), (5.11) and (5.12) that

$$\left| \int \xi d\nu_n - \int \xi dY_n \right| \leq \zeta_n + w_\xi(\eta \varepsilon_n) + \frac{2a_n}{b_n - a_n} \|\xi\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Hence, together with (5.8), it implies that $\nu_{n_k} \rightarrow \nu$ and thus $\nu \in V_f(\hat{x})$. Therefore, $\text{Closure}(V) \subseteq V_f(\hat{x})$.

Step 4 verifying $V_f(\hat{x}) \subseteq \text{Closure}(V)$.

Let $\nu \in V_f(\hat{x})$ be given. There exists a sequence $n_k \uparrow \infty$ such that $\nu_{n_k} \rightarrow \nu$. Let $\varepsilon > 0$ and $\xi \in \{\varphi_j\}_{j=1}^\infty = \cup_{n \geq 1} F_n$ be given. For fixed n_k , let $i = i(n_k)$ be the largest integer such that $b_{i-1} \leq n_k$. Let n_k (and hence i) be so large that

$$w_\xi(2^{-i+1}) < w_\xi(2^{-i+2}) < \frac{\varepsilon}{4}.$$

Let $\alpha = 1$ if $b_{i-1} \leq n_k \leq a_i$. Otherwise, $a_i < n_k \leq b_i$. Write $n_k = a_i + mp_i + l$, $0 \leq l < p_i$ and define

$$\alpha = (b_{i-1} - a_{i-1})(b_{i-1} - a_{i-1} + n_k - a_i - l)^{-1}.$$

Recall

$$\int \xi d\nu_{n_k} = \frac{1}{n_k} \sum_{j=0}^{n_k-1} \xi(f^j \hat{x}).$$

Using the inequality (5.10) again, with $A = [a_{i-1}, b_{i-1}) \cup [a_i, n_k - l)$, one obtain

$$\begin{aligned}
& \left| \int \xi d\nu_{n_k} - \frac{1}{b_{i-1} - a_{i-1} + n_k - a_i - l} \left(\sum_{j=a_{i-1}}^{b_{i-1}-1} \xi(f^j \hat{x}) + \sum_{j=a_i}^{n_k-1-l} \xi(f^j \hat{x}) \right) \right| \\
& \leq 2(l + a_{i-1} + a_i - b_{i-1})(b_{i-1} - a_{i-1} + n_k - a_i - l)^{-1} \|\xi\| \\
& \leq 2 \left(\frac{p_i}{b_{i-1} - a_{i-1}} + \frac{a_{i-1}}{b_{i-1} - a_{i-1}} + \frac{M_i}{b_{i-1} - a_{i-1}} \right) \|\xi\| \\
& \leq 2 \left(\frac{p_i}{\bar{b}_{i-1} - \bar{a}_{i-1}} + \frac{\bar{a}_{i-1}}{\bar{b}_{i-1} - \bar{a}_{i-1}} + \frac{M_i}{\bar{b}_{i-1} - \bar{a}_{i-1}} \right) \|\xi\| \\
& \leq \varepsilon \|\xi\|
\end{aligned} \tag{5.14}$$

provided n_k are large enough due to Remark 5.2.

Remark 5.3. In [9], Sigmund defined

$$a_0 = b_0 = 0$$

and

$$a_i = b_{i-1} + M_i, \quad b_i = a_i + 2^i(a_i + M_{i+1})p_i, \quad i \in \mathbb{N}.$$

It is obvious that these b_{i-1} and a_i were chosen independent of p_i . Here, in our definition (before (5.7)), the choice of b_{i-1} and a_i are chosen much larger not only than a_{i-1}, M_i, p_{i-1} but also than p_i . This is one of the important differences to Sigmund's proof. In fact, the assumption of

$$n_k - a_i = mp_i$$

in Step 4 in Sigmund's proof is not suitable. The remainder ℓ is not greater than p_i . However, in his proof, the period p_i may not be small comparing with the lap $b_{i-1} - a_{i-1}$ and hence that ℓ is not small enough with respect to $b_{i-1} - a_{i-1}$, which is necessary to the proof as shown in the above inequality (5.14). \square

Then inequality (5.14) implies that

$$\left| \int \xi d\nu_{n_k} - \left[\alpha \frac{1}{b_{i-1} - a_{i-1}} \sum_{j=a_{i-1}}^{b_{i-1}-1} \xi(f^j \hat{x}) + (1 - \alpha) \frac{1}{n_k - a_i - l} \sum_{j=a_i}^{n_k-1-l} \xi(f^j \hat{x}) \right] \right| \leq \varepsilon \|\xi\|.$$

Set

$$\rho_{n_k} = \alpha Y_{i-1} + (1 - \alpha) Y_i.$$

Using inequality (5.12), one has

$$\left| \int \xi d\nu_{n_k} - \int \xi d\rho_{n_k} \right| \leq 2\varepsilon \|\xi\|$$

for k large enough such that $n_k \gg n_\varepsilon$. Thus ρ_{n_k} has the same limit as ν_{n_k} , that is, ν .

On the other hand, the limit of ρ_{n_k} has to be in $Closure(V)$, since

$$\tilde{d}(\rho_{n_k}, V) \leq \tilde{d}(\rho_{n_k}, Y_i) \leq \tilde{d}(Y_{i-1}, Y_i) \leq 2\zeta_{i-1} + 2\zeta_i$$

and $\zeta_i \downarrow 0$. Hence, $\nu \in Closure(V)$.

The arbitrariness of $x_* \in \tilde{\Lambda}$ and δ implies the density of \hat{x} in $\tilde{\Lambda}$. Note that $\tilde{\Lambda} \subseteq supp(\omega)$ and $\omega(\tilde{\Lambda}) = 1$ and ω is an ergodic measure. All these conditions ensure that $Closure(\tilde{\Lambda}) = supp(\omega)$. Hence, it holds that such \hat{x} are dense in $supp(\omega)$. This ends the whole proof. \square

Remark 5.4. Note that $\mathcal{M}_{inv}(\tilde{\Lambda})$ is convex but may not be compact. For better understanding Theorem 1.1, here we construct a compact connected subset of $\mathcal{M}_{inv}(\tilde{\Lambda})$. Let $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots)$ be a (weak) decreasing sequence of positive real numbers which approach zero. Let

$$\mathcal{M}_{\bar{\varepsilon}} = \{\nu \in \mathcal{M}_{inv}(f) : \nu(\tilde{\Lambda}_\ell) \geq 1 - \varepsilon_\ell, \ell = 1, 2, \dots\}.$$

Since each $\tilde{\Lambda}_\ell$ is compact, the map $\nu \rightarrow \nu(\tilde{\Lambda}_\ell)$ is upper-semicontinuous. Hence, $\mathcal{M}_{\bar{\varepsilon}}$ is a closed convex subset of $\mathcal{M}_{inv}(f)$, the set of all the invariant measures of f . This implies $\mathcal{M}_{\bar{\varepsilon}}$ is a compact connected subset of $\mathcal{M}_{inv}(f)$. Since every $\nu \in \mathcal{M}_{\bar{\varepsilon}}$ satisfying $\nu(\tilde{\Lambda}) = 1$, we can regard $\mathcal{M}_{\bar{\varepsilon}}$ as a subset of $\mathcal{M}_{inv}(\tilde{\Lambda})$. Thus, $\mathcal{M}_{\bar{\varepsilon}}$ must be a compact connected subset of $\mathcal{M}_{inv}(\tilde{\Lambda})$. \square

6 Proof of Theorem 1.3 and 1.5

In this section, we use Theorem 1.1 to prove Theorem 1.3 and then use Theorem 1.3 to prove Theorem 1.5.

Proof of Theorem 1.3 The proof is not difficult and analogical with the proof of Proposition 21.18 in [2]. Since $\mathcal{M}(f)$ is compact and convex, we can find open balls B_n, C_n in $\mathcal{M}(f)$ such that

- (a). $B_n \subset Closure(B_n) \subset C_n$;
- (b). $diam C_n \rightarrow 0$;
- (c). $B_n \cap Closure\{\nu \in \mathcal{M}_{inv}(M) \mid \nu(\tilde{\Lambda}) = 1\} \neq \emptyset$;
- (d). each point of $Closure\{\nu \in \mathcal{M}_{inv}(M) \mid \nu(\tilde{\Lambda}) = 1\}$ lies in infinitely many B_n .

Put

$$P(C_n) = \{x \in M \mid V_f(x) \cap C_n \neq \emptyset\}, \quad \forall n \in \mathbb{Z}^+.$$

It can be verified that the set of points with maximal oscillation is just $\cap_{n \geq 1} P(C_n)$. Note that

$$\begin{aligned} P(C_n) &\supseteq \{x \in M \mid \forall N_0 \in \mathbb{Z}^+, \exists N > N_0 \text{ with } \delta(x)^N \in B_n\} \\ &= \cap_{N_0=1}^{\infty} \cup_{N > N_0} \{x \in \tilde{\Lambda} \mid \delta(x)^N \in B_n\} \end{aligned} \quad (6.15)$$

Since $x \rightarrow \delta(x)^N$ is continuous (for fixed N), the sets $\cup_{N > N_0} \{x \in M \mid \delta(x)^N \in B_n\}$ are open. Since $B_n \cap \text{Closure}\{\nu \in \mathcal{M}_{inv}(M) \mid \nu(\tilde{\Lambda}) = 1\} \neq \emptyset$, these sets are also dense, as shown in Corollary 1.2. Hence $\cap_{n \geq 1} P(C_n)$ contains a dense G_δ -set. \square

Proof of Theorem 1.5 Let x be a point having maximal oscillation. By assumption there at least exist two invariant measures $\mu_1 \neq \mu_2 \in V_f(x)$. So there is continuous function ϕ such that

$$\int \phi d\mu_1 \neq \int \phi d\mu_2. \quad (6.16)$$

Due to the definition of $V_f(x)$, there are two sequences of integers $n_k, m_k \rightarrow +\infty$ such that

$$\delta^{n_k}(x) \rightarrow \mu_1, \quad \delta^{m_k}(x) \rightarrow \mu_2. \quad (6.17)$$

These imply that

$$\lim_{n_k} \frac{1}{n_k} \sum_{j=0}^{n_k} \phi(f^j x) = \int \phi d\mu_1 \text{ and } \lim_{m_k} \frac{1}{m_k} \sum_{j=0}^{m_k} \phi(f^j x) = \int \phi d\mu_2.$$

Combining these equalities with (6.16), we can deduce that

$$\lim_{n_k} \frac{1}{n_k} \sum_{j=0}^{n_k} \phi(f^j x) = \int \phi d\mu_1 \neq \int \phi d\mu_2 = \lim_{m_k} \frac{1}{m_k} \sum_{j=0}^{m_k} \phi(f^j x).$$

Thus we have that $\lim_{n_k} \frac{1}{n_k} \sum_{j=0}^{n_k} \phi(f^j x)$ does not exist. \square

Acknowledgement. The authors thank very much to the whole seminar of dynamical systems in Peking University.

References.

1. R. Bowen, Entropy for group endomorphisms and homogeneous spaces, *Trans. Amer. Math. Soc.* 153(1971), 401-414.
2. M. Denker, C. Grillenberger, K. Sigmund, Ergodic Theory on the Compact Space, *Lecture Notes in Mathematics* **527**.
3. M. Hirayama, Periodic probability measures are dense in the set of invariant measures, *Dist. Cont. Dyn. Sys.* 9 (2003), 1185-1192.

4. A. Katok, *Liapunov exponents, entropy and periodic orbits for diffeomorphisms*, Pub. Math. IHES, 51 (1980) 137-173.
5. C. Liang, G. Liu, W. Sun, *Approximation properties on invariant measure and Oseledec splitting in non-uniformly hyperbolic systems*, Trans. Amer. Math. Soci. 361 (2009) 1543-1579
6. V. I. Oseledec, *Multiplicative ergodic theorem, Liapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc., 19 (1968), 197-221; translated from Russian.
7. M. Pollicott, *Lectures on ergodic theory and Pesin theory on compact manifolds*, Cambridge Univ. Press, 1993
8. C. Pugh, *The $C^{1+\alpha}$ hypothesis in Pesin theory*, (English) [J] Publ. Math., Inst. Hautes tud. Sci. 59, 143-161 (1984).
9. K. Sigmund, *Generic properties of invariant measures for Axiom A-diffeomorphisms*, Invent.Math. 11 (1970),99-109.
10. W. Sun, X. Tian, *Pesin set, closing lemma and shadowing lemma in C^1 non-uniformly hyperbolic systems with limit domination*, arXiv:1004.0486.
11. P. Walters, *An introduction to ergodic theory*, Springer-Verlag, 2001.